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An Introduction to Tensors for Students of Physics and Engineering

Joseph C. Kolecki
Glenn Research Center, Cleveland, Ohio

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An Introduction To Tensors for Students of Physics and Engineering

Joseph C. Kolecki
National Aeronautics and Space Administration
Glenn Research Center
Cleveland, Ohio 44135

Tensor analysis is the type of subject that can make even the best of students shudder. My own post-graduate instructor in the subject took away much of the fear by speaking of an implicit *rhythm* in the peculiar notation traditionally used, and helped me to see how this rhythm plays its way throughout the various formalisms.

Prior to taking that class, I had spent many years "playing" on my own with tensors. I found the going to be tremendously difficult, but was able, over time, to back out some physical and geometrical considerations that helped to make the subject a little more transparent. Today, it is sometimes hard not to think in terms of tensors and their associated concepts.

This article, prompted and greatly enhanced by Marlos Jacob, whom I've met only by e-mail, is an attempt to record those early notions concerning tensors. It is intended to serve as a bridge from the point where most undergraduate students "leave off" in their studies of mathematics to the place where most texts on tensor analysis begin. A basic knowledge of vectors, matrices, and physics is assumed. A semi-intuitive approach to those notions underlying tensor analysis is given via scalars, vectors, dyads, triads, and similar higher-order vector products. The reader must be prepared to do some mathematics and to think.

For those students who wish to go beyond this humble start, I can only recommend my professor's wisdom: find the *rhythm* in the mathematics and you will fare pretty well.

Beginnings

At the heart of all mathematics are numbers.

If I were to ask how many marbles you had in a bag, you might answer, "Three." I would find your answer perfectly satisfactory. The 'bare' number 3, a magnitude, is sufficient to provide the information I seek.

If I were to ask, "How far is it to your house?" and you answered, "Three," however, I would look at you quizzically and ask, "Three *what*?" Evidently, for this question, more information is required. The bare number 3 is no longer sufficient; I require a '*denominate*' number – a number with a name.

Suppose you rejoindered, "Three km." The number 3 is now *named* as representing a certain number of *km*. Such numbers are sometimes called *scalars*. Temperature is represented by a scalar. The total energy of a thermodynamic system is also represented by a scalar.

If I were next to ask "Then how do I get to your house from here?" and you said, "Just walk three km," again I would look at you quizzically. This time, not even a denominate number is sufficient; it is necessary to specify a distance or magnitude, yes, but in which direction?

"Just walk three km *due north*." The denominate number *3 km* now has the required additional *directional* information attached to it. Such numbers are called *vectors*. Velocity is a vector since it has a magnitude and a direction; so is momentum. Quite often, a vector is represented by *components*. If you were to tell me that to go from here to your house I must walk three blocks east, two blocks north, and go up three floors, the vector extending from "here" to "your house" would have three spatial components:

- Three blocks east,
- Two blocks north,
- Three floors up.

Physically, vectors are used to represent locations, velocities, accelerations, flux densities, field quantities, etc. The defining equations of the gravitational field in classical dynamics (Newton's Law of Universal Gravitation), and of the electromagnetic field in classical electrodynamics (Maxwell's four equations) are all given in vector form. Since vectors are higher order quantities than scalars, the physical realities they correspond to are typically more complex than those represented by scalars.

A Closer Look at Vectors

The action of a vector is equal to the sum of the actions of its components. Thus, in the example given above, the vector from "here" to "your house" can be represented as

$$\mathbf{V} =^{[1]} 3 \text{ blocks east} + 2 \text{ blocks north} + 3 \text{ floors up}$$

Each component of **V** contains a vector and a scalar part. The scalar and vector components of **V** can be represented as follows:

- *Scalar*: Let a = 3 blocks, b = 2 blocks, and c = 3 floors be the scalar components; and
- *Vector*: Let **i** be a unit vector pointing east, **j** be a unit vector pointing north, and **k** be a *unit vector* pointing up. (N.B.: Unit vectors are non-denominate, have a magnitude of unity, and are used only to specify a direction.)

Then the total vector, in terms of its scalar components and the unit vectors, can be written as

$$\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

This notation is standard in all books on physics and engineering. It is also used in books on introductory mathematics.

Next, let us look at how vectors combine. First of all, we know that numbers may be combined in various ways to produce new numbers. For example, six is the *sum* of three and three or the *product* of two and three. A similar logic holds for vectors. Vector rules of combination include vector addition, scalar (dot or inner) multiplication, and (in three dimensions) cross multiplication. Two vectors, **U** and **V**, can be added to produce a new vector **W**:

$$\mathbf{W} = \mathbf{U} + \mathbf{V}.$$

[1] The appropriate symbol to use here is "⇒" rather than "=" since the 'equation' is not a strict vector identity. However, for the sake of clarity, the "⇒" notation has been suppressed both here and later on, and "=" signs have been used throughout. There is no essential loss in rigor, and the meaning should be clear to all readers.

Vector addition is often pictorially represented by the so-called *parallelogram rule*. This rule is a pencil and straightedge construction that is strictly applicable only for vectors in Euclidean space, or for vectors in a curved space *embedded in a Euclidean space of higher dimension*, where the parallelogram rule is applied in the higher dimensional Euclidean space. For example, two tangent vectors on the surface of a sphere may be combined via the parallelogram rule provided that the vectors are represented in the Euclidean 3-space which contains the sphere. In formal tensor analysis, such devices as the parallelogram rule are generally not considered.

Two vectors, **U** and **V** can also be combined via an inner product to form a new scalar η. Thus

$$\mathbf{U} \cdot \mathbf{V} = \eta.$$

Example: The inner product of *force* and *velocity* gives the scalar *power* being delivered into (or being taken out of) a system:

$$\mathbf{f}(\text{nt}) \cdot \mathbf{v}(\text{m/s}) = \text{p(W)}.$$

Example: The inner product of a vector with itself is the square of the magnitude (length) of the vector:

$$\mathbf{U} \cdot \mathbf{U} = U^2.$$

Two vectors **U** and **V** in three-dimensional space can be combined via a cross product to form a new (axial) vector:

$$\mathbf{U} \times \mathbf{V} = \mathbf{S}$$

where **S** is perpendicular to the plane containing **U** and **V** and has a sense (direction) given by the *right-hand rule*.

Example: Angular momentum is the cross product of linear momentum and distance:

$$\mathbf{p}(\text{kg m/s}) \times \mathbf{s}(\text{m}) = \mathbf{L}(\text{kg m}^2/\text{s}).$$

Finally, a given vector **V** can be multiplied by a scalar number α to produce a new vector with a different magnitude but the same direction. Let $\mathbf{V} = V\mathbf{u}$ where **u** is a unit vector. Then

$$\alpha\mathbf{V} = \alpha V\mathbf{u} = (\alpha V)\mathbf{u} = \xi\mathbf{u}$$

where ξ is the new magnitude.

Example: Force (a vector) equals mass (a scalar) times acceleration (a vector):

$$\mathbf{f}(\text{nt}) = \text{m(kg)}\ \mathbf{a}(\text{m/s}^2)$$

where the force and the acceleration share a common direction.

Introducing Tensors: Magnetic Permeability and Material Stress

We have just seen that vectors can be multiplied by scalars to produce new vectors with the same sense or direction. In general, we can specify a unit vector **u**, at any location we wish, to point in any direction we please. In order to construct another vector from the unit vector, we multiply **u** by a scalar, for example λ, to obtain $\lambda\mathbf{u}$, a new vector with magnitude λ and the sense or direction of **u**.

Notice that the effect of multiplying the unit vector by the scalar is *to change the magnitude from unity to something else, but to leave the direction unchanged*. Suppose we wished to alter both the magnitude *and* the direction of a given vector. Multiplication by a scalar is no longer sufficient. Forming the cross product with another vector is also not sufficient, unless we wish to limit the change in direction to right angles. We must find and use another kind of mathematical 'entity.'

Let's pause to introduce some terminology. We will rename the familiar quantities of the previous paragraphs in the following way:

- Scalar: *Tensor of rank 0*. (magnitude only – 1 component)
- Vector: *Tensor of rank 1*. (magnitude and one direction – 3 components)

This terminology is suggestive. Why stop at rank 1? Why not go onto rank 2, rank 3, and so on.

- Dyad: *Tensor of rank 2*. (magnitude and two directions – $3^2 = 9$ components)
- Triad: *Tensor of rank 3*. (magnitude and three directions – $3^3 = 27$ components)
- Etcetera…

We will now merely state that *if we form the inner product of a vector and a tensor of rank 2,a dyad, the result will be another vector with both a new magnitude and a new direction*. (We will consider triads and higher order objects later.)

A tensor of rank 2 is defined as a system that has a magnitude and two directions associated with it. It has 9 components. For now, we will use an example from classical electrodynamics to illustrate the point just made.

The magnetic flux density **B** in volt-sec/m^2 and the magnetization **H** in Amp/m are related through the permeability μ in H/m by the expression

$$\mathbf{B} = \mu\mathbf{H}.$$

For free space, μ is a scalar with value $\mu\ (= \mu_0) = 4\pi \times 10^{-7}$ H/m. Since μ is a scalar, the flux density and the magnetization in free space differ in magnitude but not in direction. In some exotic materials, however, the component atoms or molecules have peculiar dipole properties that make these terms differ in both magnitude *and* direction. In such materials, the scalar permeability is then replaced by the tensor permeability $\underline{\boldsymbol{\mu}}$, and we write, in place of the above equation,

$$\mathbf{B} = \underline{\boldsymbol{\mu}} \cdot \mathbf{H}.$$

The permeability $\underline{\boldsymbol{\mu}}$ is a tensor of rank 2. Remember that **B** and **H** are both vectors, but they now differ from one another in both magnitude *and* direction.

The classical example of the use of tensors in physics has to do with stress in a material object. Stress has the units of *force-per-unit-area*, or nt/m^2. It seems clear, therefore, that (stress) × (area) should equal (force); i.e., the stress-area product should be associated with the applied forces that are producing the stress. We know that force is a vector. We also know that area can be represented as a vector by associating it with a direction, i.e., the differential area d**S** is a vector with magnitude dS and direction normal to the area element, pointing outward from the convex side.

Thus, the *stress* in the equation (force) = (stress) × (area) must be either a scalar or a tensor. If stress were a scalar, then a single denominate number should suffice to represent the stress at any point within a material. But an immediate problem arises in that there are two different types of stress: tensile stress (normal force) and shear stress (tangential force). How can a single denominate number represent both? Additionally, stresses have directional properties more like "vector times vector" (or *dyad*) than simply "vector." We must conclude that stress is a tensor – it is, in fact, another tensor of rank 2 – and that the force must be an inner product of stress and area.

The force d**F** due to the stress $\underline{\mathbf{T}}$ acting on a differential surface element d**S** is thus given by

$$d\mathbf{F} = \underline{\mathbf{T}} \cdot d\mathbf{S}.$$

The right-hand side can be integrated over any surface within the material under consideration, as is actually done, for example, in the analysis of bending moments in beams. The stress tensor $\underline{\mathbf{T}}$ was the first tensor to be described and used by scientists and engineers. The word *tensor* derives from the Latin *tensus* meaning stress or tension.

In summary, notice that in the progression from single number to scalar to vector to tensor, etc., information is being added at every step. The complexity of the physical situation being modeled determines the rank of the tensor representation we must choose. A tensor of rank 0 is sufficient to represent a single temperature or a temperature field across a surface, for example, an aircraft compressor blade. A tensor of rank 1 is required to represent the electric field surrounding a point charge in space or the gravitational field of a massive object. A tensor of rank 2 is necessary to represent a magnetic permeability in complex materials, or the stresses in a material object or in a field, and so on...

Preliminary Mathematical Considerations

Let's consider the *dyad* – the "vector times vector" product mentioned above – in a little more detail. Dyad products were the mathematical precursors to actual tensors, and, although they are somewhat more cumbersome to use, their relationship with the physical world is somewhat more intuitive because they directly build from more traditional vector concepts understood by physicists and engineers.

In constructing a dyad product from two vectors, we form the term-by-term product of each of their individual components and add. If **U** and **V** are the two vectors under consideration, their *dyad product* is simply **UV**. The dyad product **UV** is *neither a dot nor a cross product*. It is a distinct entity unto itself. If $\mathbf{U} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{V} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{UV} = u_1v_1\mathbf{ii} + u_1v_2\mathbf{ij} + u_1v_3\mathbf{ik} + u_2v_1\mathbf{ji} + \cdots$$

where **i**, **j**, and **k** are unit vectors in the usual sense and **ii**, **ij**, **ik**, etc. are unit dyads. In forming the product **UV** above, we simply "did what came naturally" (a favorite phrase of another of my professors!) from our knowledge of multiplying polynomials in elementary algebra. Notice that, by setting $u_1v_1 = \mu_{11}$, $u_1v_2 = \mu_{12}$, etc., this dyad can be rewritten as

$$\mathbf{UV} = \mu_{11}\mathbf{ii} + \mu_{12}\mathbf{ij} + \mu_{13}\mathbf{ik} + \mu_{21}\mathbf{ji} + \cdots$$

and that the scalar components μ_{ij} can be arranged in the familiar configuration of a 3x3 *matrix*:

$$\begin{matrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \end{matrix}$$

μ_{31} μ_{32} μ_{33}

All dyads can have their scalar components represented as matrices. Just as a given matrix is generally not equal to its transpose, so with dyads it is generally the case that $\mathbf{UV} \neq \mathbf{VU}$, i.e., the dyad product is not commutative.

We know that a matrix can be multiplied by another matrix or by a vector. We also know that, given a matrix, the results of pre- and post-multiplication are usually different; i.e., matrix multiplication does not, in general, commute. This property of matrices is used extensively in the "bra-" and "ket-" formalisms of quantum mechanics.

Using the known rules of matrix multiplication, we can, by extension, write the rules associated with dyad multiplication.

The product of a matrix $\underline{\mathbf{M}}$ and a scalar α is commutative. Let the scalar components of $\underline{\mathbf{M}}$ be represented by the 3×3 matrix $[\mu_{ij}]$ i, j = 1, 2, 3; (i.e., the scalar components of $\underline{\mathbf{M}}$ can be thought of as the same array of numbers shown above). Then for any scalar α, we find

$$\alpha\underline{\mathbf{M}} = [\alpha\mu_{ij}] = [\mu_{ij}\alpha] = \underline{\mathbf{M}}\alpha.$$

Similarly, the product of a dyad **UV** and a scalar α is defined as

$$\alpha(\mathbf{UV}) = (\alpha\mathbf{U})\mathbf{V} = (\mathbf{U}\alpha)\mathbf{V} = \mathbf{U}(\alpha\mathbf{V}) = \mathbf{U}(\mathbf{V}\alpha) = (\mathbf{UV})\alpha.$$

In this case, the results of pre- and post-multiplication are equal.

The inner product of a matrix and a vector, however, is not commutative. Let $\mathbf{V} \Rightarrow (V_i)$ be a row vector with i = 1, 2, 3, and $\underline{\mathbf{M}} = [\mu_{ij}]$ as before. Then, when we pre-multiply,

$$\mathcal{U} = \mathbf{V} \cdot \underline{\mathbf{M}} = (\mathcal{U}_j) = [\Sigma_i V_i \mu_{ij}]$$

where the summation is over the first matrix index i.

When we post-multiply with $\mathbf{V} = (V_j)$ now re-arranged as a column vector,

$$\mathcal{U}^* = \underline{\mathbf{M}} \cdot \mathbf{V} = (\mathcal{U}^*_i) = [\Sigma_j \mu_{ij} V_j]$$

where the summation is over the second matrix index j. It is clear that $\mathcal{U}^* \neq \mathcal{U}$.

Similarly, the inner product of the dyad **UV** with another vector **S** is defined to be

$$\mathbf{S} \cdot (\mathbf{UV})$$

when we pre-multiply, and

$$(\mathbf{UV}) \cdot \mathbf{S}$$

when we post-multiply. As with matrices, pre- and post-multiplication *do* make a difference to the resulting object. To maintain consistency with matrix-vector multiplication, the dot "attaches" as follows:

$$\mathbf{S} \cdot \mathbf{UV} = (\mathbf{S} \cdot \mathbf{U})\mathbf{V} = \sigma\mathbf{V}$$

where $\sigma = \mathbf{S} \cdot \mathbf{U}$. The result is a vector with magnitude σ and sense (direction) determined by **V**. But

$$\mathbf{UV \cdot S = U(V \cdot S) = U\lambda = \lambda U}$$

is a vector with magnitude λ and sense determined by **U**. It should be clear that, in general,

$$\mathbf{S \cdot UV \neq UV \cdot S}.$$

Tensors of Rank > 2

Tensors of rank 2 result from dyad products of vectors. In an entirely analogous way, tensors of rank 3 arise from triad products, **UVW**, and tensors of rank n arise from "n-ad" products of vectors, **UVW...AB**. In three-dimensional space, the number of components in each of these systems is 3^n. The rules governing these higher rank objects are defined in the same way as the rules developed above.

Specific Statements for Tensors *per se*

We now extend the properties and rules of operation for familiar objects – scalars, vectors, and matrices – to tensors *per se*. We will summarize our previous remarks in Items 1–5 and then make definitive statements explicitly for tensors in Items 6–12.

1. All *scalars* are not tensors, although all tensors of rank 0 are scalars (see below).
2. All *vectors* are not tensors, although all tensors of rank 1 are vectors (see below).
3. All *dyads* or *matrices* are not tensors, although all tensors of rank 2 are dyads or matrices.
4. We have examined, in some detail, properties and operating rules for scalars, vectors, dyads, and matrices.
5. We now *extend these rules to tensors per se*. We assert that:
6. Tensors can be multiplied by other tensors to form new tensors.
7. The product of a tensor and a scalar (tensor of rank 0) is commutative.
8. The pre-multiplication of a given tensor by another tensor produces a different result from post-multiplication; i.e., tensor multiplication in general is *not* commutative.
9. The rank of a new tensor formed by the product of two other tensors is the sum of their individual ranks.
10. The inner product of a tensor and a vector or of two tensors is not commutative.
11. The rank of a new tensor formed by the inner product of two other tensors is the sum of their individual ranks minus 2.
12. A tensor of rank n in three-dimensional space has 3^n components.

Re-examining Magnetic Permeability and Material Stress

Now we can see why the magnetic permeability (the stress in a material object) must be a rank 2 tensor. In this section and the next, we will use a more formal approach, proceeding more in the style of a mathematician writing a proof than a physicist or engineer solving a problem. While the approach is formal, the conclusions are physically as well as mathematically valid.

Let's begin with the magnetic field. We use the tensor form

$$\mathbf{B = \underline{\mu} \cdot H}$$

and represent the tensor permeability by a dyad $\underline{\mu}$ **= UV** *without concern for the specific natures of the vectors* **U** *and* **V.** This step is perfectly valid. Even though we cannot specify the exact

physical natures of **U** and **V**, we understand that a second rank tensor and a dyad are equivalent provided the vectors **U** and **V** are appropriately chosen. We make this assumption without loss of generality. From the physicist/engineer perspective, it is only significant that

- the dyad **UV** represents a physical quantity – i.e., permeability $\underline{\boldsymbol{\mu}}$; and
- the rules developed in the previous section can be applied to the dyad representation in a strictly formalistic way to advance the argument at hand.

With these ideas in mind, we write

$$\mathbf{B} = \mathbf{UV} \cdot \mathbf{H} = \mathbf{U}(\mathbf{V} \cdot \mathbf{H}) = \mathbf{U}\lambda = \lambda\mathbf{U}$$

where $\lambda = \mathbf{V} \cdot \mathbf{H}$ is a scalar and **U** a vector. It is clear that the direction of **B** depends only on the direction of **U,** not **H**. Since we specified nothing about the nature of **U**, **U** cannot be restricted in its magnitude or direction by **H** in any way. Therefore, we conclude that the direction of **B** must be independent of the direction of **H**.

In tensor (or, in this case, matrix) notation, we might represent the scalar components of the magnetization vector by H_i and the permeability scalar components by a Cartesian matrix μ_{st}. The flux density then becomes a Cartesian vector whose scalar components are B_s where

$$B_s = \Sigma_t \, \mu_{st} \, H_t$$

with summation occurring over the repeated index, t. This last representation has become the standard in the literature.

Similarly, if we represent material stress as a dyad, for example, $\underline{\mathbf{T}} = \mathbf{UV}$ (again, without concern for the specific nature of **U** and **V**), then the inner product $\underline{\mathbf{T}} \cdot d\mathbf{S}$, can be represented as

$$\underline{\mathbf{T}} \cdot d\mathbf{S} = (\mathbf{UV}) \cdot d\mathbf{S} = \mathbf{U}(\mathbf{V} \cdot d\mathbf{S}) = \mathbf{U}\, d\zeta$$

where $d\zeta$ is the scalar differential resulting from the inner product $\mathbf{V} \cdot d\mathbf{S}$. The term $\mathbf{U}\, d\zeta$ is a vector (tensor of rank 1) and is, in fact, the differential force $d\mathbf{F}$ acting on the area element

$$\mathbf{U}\, d\zeta = d\mathbf{F}.$$

Changing Tensor Rank: Contraction

It is inevitable in an article of this type that we must do some mathematics. The previous section used a more formal (less intuitive) approach to demonstrate one role that tensors play in physics and engineering. In this section, we will stay with the formal approach and define yet another, perhaps somewhat peculiar, tensor operation, which will be left without much physical (intuitive) consideration. The student, who so wishes, can skip over this section without loss.

We begin by summarizing the relationship between the type of vector product being used and the rank of the resulting object. These results are already implicit in the material given above.

- A vector-scalar product results in a vector: there is no change in rank.
- A vector-vector dyad product results in a dyad: there is an increase in rank from rank 1 (vector) to rank 2.
- A vector-vector inner product usually results in a scalar: there is a decrease in rank from rank 1 (vector) to rank 0 (scalar).
- And so on...

Except for the inner product, the rank of the resulting quantity is the sum of the ranks of the quantities being combined. So, if we form a triad **UVW**, its components comprise a tensor of rank 3. If we form a "tetrad," its components comprise a tensor of rank 4. And so on...

Now it is time to introduce that "somewhat peculiar" new rule. Please rest assured that its practical usefulness is fully attested in all of the advanced works in physics and engineering where tensors are applied.

Let's say that we have a vector "n-ad,"

$$\underbrace{\mathbf{UVW...ABC}}_{\text{n vectors}}$$

We can form a new tensor by introducing a dot between *any two consecutive* terms. Thus, we might choose to introduce a dot between **U** and **V**, or **V** and **W**, etc. This process is called *contraction*, and results in a new tensor with rank (n – 2).

- If we introduce a dot into an existing dyad, the dyad is *contracted* to a scalar. Thus, given the dyad **UV**, we can introduce the dot forming a new tensor $\mathbf{U} \cdot \mathbf{V}$, which is a scalar. And there is a reduction in rank by two. A special case of interest is the dyad **UU**. Contraction of the dyad gives the squared magnitude of the vector **U**: $\mathbf{U} \cdot \mathbf{U} = U^2$.
- If we introduce a dot into an existing triad, the triad is *contracted* to a vector. Given the triad **UVW**, we can introduce a dot in one of two ways, forming either $\mathbf{U} \cdot \mathbf{VW}$ or $\mathbf{UV} \cdot \mathbf{W}$, either of which is a vector, since

$$\mathbf{U} \cdot \mathbf{VW} = (\mathbf{U} \cdot \mathbf{V})\mathbf{W} = \alpha\mathbf{W} \text{ where } (\alpha = \mathbf{U} \cdot \mathbf{V})$$

or

$$\mathbf{UV} \cdot \mathbf{W} = \mathbf{U}(\mathbf{V} \cdot \mathbf{W}) = (\mathbf{V} \cdot \mathbf{W})\mathbf{U} = \beta\mathbf{U} \text{ where } (\beta = \mathbf{V} \cdot \mathbf{W}).$$

Notice that the two results are different – depending on the placement of the dot.

Contraction of a tensor of rank (n) always results in another tensor of rank (n – 2). If we were to form the force-velocity dyad **FV**, as might be done in formulating the general equations of fluid dynamics, we could always find the rate of energy dissipated in the fluid (the power) by contracting the dyad to a scalar. Thus

$$dE/dt = \mathbf{F} \cdot \mathbf{V}.$$

Essentially, given a tensor equation of rank n, it is possible to extract information from the equation in a variety of ways. The ability of tensor equations both to store information and to permit its simple manipulation should be coming clear by now!

Invariance of Physical Quantities: Introducing Coordinate Transformations

Tensors are typically defined by their coordinate transformation properties. The transformation properties of tensors can be understood by realizing that the physical quantities they represent must appear in certain ways to different observers with different points of view.

Suppose, for example, that I measure the temperature (°C) at a given point P at a given time. You also measure the temperature (°C) at P at the same time but from a different location that is in motion relative to my location. Would it make any sense if you and I acquired different

magnitudes; i.e., if my thermometer measured 25°C and yours measured 125°C? No. We must both obtain the same quantity from our respective measurements.

Put another way, suppose that I call my point of view (coordinate system or reference frame) K and yours K*. Let T be the temperature (°C) measured at P in K and T* be the temperature (°C) measured in K*. We then *require*

$$T = T^*.$$

This expression is an example of a coordinate transformation law between K and K* for the scalar temperature T. *Only scalars that transform like this are to be admitted into the class of tensors of rank 0*. In fact, letting T stand for any scalar quantity we wish, the equation T = T* can be taken as the *definition* of a tensor of rank 0.

Now let T be the frequency of light emanating from a monochromatic source at P. Again, let two observers, K and K*, measure the frequency of the light at P at the same time using the same units of inverse seconds. If I am stationary relative to the source, the light will have a certain frequency, for example $T = \upsilon_0$. If, on the other hand, *you* are moving toward or away from the source when you take your measurement, the light will be red or blue shifted with frequency $T^* = \upsilon_0 \pm \Delta\upsilon$. Obviously $T \neq T^*$ in this case, and although the frequency thus observed is a scalar, it is evidently, *not* a tensor of rank 0.

A similar argument holds for vectors. As was the case with scalars, not all vectors are tensors of rank 1. Suppose that a vector quantity **V** exists at a point P. Again, assume two reference frames, K and K*. Let **V** be the vector observed (measured) in K, and **V*** be the same vector observed in K* at the same time. As with the temperature example, we again require that

$$\mathbf{V} = \mathbf{V}^*$$

since, after all, K and K* are both observing the self-same vector. Any other result would not make physical sense. Any vector that transforms according to the expression **V** = **V*** is *defined* to be a tensor of rank 1. We usually say that the transformation law **T** = **T***, or **V** = **V***, requires the quantity represented by **T** or **V** to be *coordinate independent*.

While the vector itself is coordinate independent, its individual components are not. Thus, in the vector transformation law **V** = **V***, the components of the vector vary from system to system, but do so in such a way that the vector quantity itself remains unchanged. This truth is evident when we realize that the components in any coordinate system are nothing more than the projections of the vector onto the local coordinate axes.

Many representations exist for vectors in Euclidean 3-space, the space of our school algebra and geometry, including the familiar $\mathbf{V} = \alpha\,\mathbf{i} + \beta\,\mathbf{j} + \gamma\,\mathbf{k}$ in which **V** is the vector being represented; α, β, and γ are its scalar components along the x, y, and z axes of a Cartesian reference system, respectively; and **i**, **j**, and **k** are unit vectors along those same axes. Another representation of **V** is as a triad of numbers, $\mathbf{V} = (\alpha, \beta, \gamma)$.

In the more general case of higher dimensional spaces, whether Euclidean or non-Euclidean, vectors are represented by a number array – a row or a column. Thus, if **V** were a vector in E_n or R_n (Euclidean or Riemannian n-space) it would be written as

$$\mathbf{V} = (v_1, v_2, \ldots, v_n)$$

or simply, $\mathbf{V} = (v_i)$, i = 1, ..., n.

Now, let **V** be the *position vector* extending from the origin of K to a particular point P, and **V*** be the position vector extending from the origin of K* to that same point. Assume that the origins of K and K* *do not coincide*; then $\mathbf{V} \neq \mathbf{V}^*$. The position vector is very definitely coordinate *dependent* and is not a tensor because it does not satisfy the condition of coordinate independence.[2]

But suppose that $\mathbf{V_1}$ and $\mathbf{V_2}$ were position vectors of points P_1 and P_2 in K, and that $\mathbf{V_1}^*$ and $\mathbf{V_2}^*$ were position vectors to the same points P_1 and P_2 in K*. The vector extending from P_1 to P_2 *must be the same vector in both systems*. This vector is $\mathbf{V_2} - \mathbf{V_1}$ in K and $\mathbf{V_2}^* - \mathbf{V_1}^*$ in K*. Thus we have

$$\mathbf{V_2} - \mathbf{V_1} = \mathbf{V_2}^* - \mathbf{V_1}^*,$$

i.e., while the position vector itself is not a tensor, the *difference* between any two position vectors *is* a tensor of rank 1! Similarly, for any position vectors V and V*, dV = dV*; i.e., the differential of the position vector is a tensor of rank 1.

This result may seem a little strange, but it provides strong motivation for exercising care in working with physical vector quantities.

A Digression: Coordinate Systems and Mathematical Spaces

Now, for one brief chapter, we are going to sidestep the main theme of this article to consider a subject that is extremely important but all too often ignored. Students who study such disciplines as General Relativity should especially appreciate the ideas introduced here.

So far, except for a few brief allusions, we have tacitly assumed that we were operating in the same Euclidean space as we encountered in our high school and college mathematics and physics without so much as a second thought as to what we were doing or why. In fact, the choice of a mathematical space – whether Euclidean or non-Euclidean – is every bit as important as the choice of a properly suited reference system from which to model physical events. In many cases, confusion exists in the minds of students, often spurred on by popular literature, regarding the distinction between coordinate systems *per se* and space. Since we are considering physical/tensorial quantities that *exist in space* and are *coordinate independent*, it behooves us to take a closer look at this distinction.

A *line* is an example of a Euclidean 1-space. It has one dimension, extends to $\pm\infty$, and has a metric (e.g.: the unit interval). The coordinate system associated with the line is defined by the unit interval, chosen for convenience then copied repeatedly, end-to-end along the entire line in both directions from the starting point. A line thus marked, with numbers added for reference, is called a *real number line*.

[2] This argument depends on the definition of the position vector as *the vector extending from the origin of a given coordinate system to a point that it is said to locate*. Thus, for any point P in space, the position vectors in two systems K and K* whose origins do not coincide will, by definition, be different. If **V** is the position vector in K, then it is also a vector in K* but not a *position* vector, and the coordinate transformations apply to it in the usual way. Since, however, **V** is a position vector only in one system, not in both, it cannot represent the same thing in both; hence, it is fundamentally different than other vector quantities whose character is the same in all reference frames.

A *plane* is an example of a Euclidean 2-space. It has two dimensions, extends to infinity in all directions, and has a linear metric (the unit interval) and an areal metric (the unit square). It also has an intrinsic geometry defined by the Greek, Euclid (c. 300 BC).

In the geometry of Euclid, objects, such as triangles, squares, or circles, can be moved about in the plane without deformation and, therefore, compared to one another using such relationships as similarity or congruence. Also in the geometry of Euclid, parallel lines extend forever without meeting, and so on. In the plane, the coordinate system of choice is the Cartesian system, comprising two real number lines that meet at right angles. Other systems are also possible.

The physical analogue of such a space is a region in which material objects and/or beams of light can be moved about without deformation. But since gravity permeates all space and time, no such region exists in the universe at large. Thus it was that Einstein abandoned Euclidean space as a basis for his General Relativity and adopted a differentially metric non-Euclidean space instead.

A *sphere* is an example of an *elliptic* 2-space. Like the plane, the sphere also has two dimensions. Unlike the plane, however, the sphere does not extend to infinity; it fact, the sphere is a closed, finite surface. The sphere has a *differential* linear metric and a *differential* areal metric. It also has a geometry, though one quite different from that of Euclid.

A differential metric is used wherever a unit metric is intractable. A unit metric on a sphere would have to be curved to fit into the surface. Such a metric, of course, could be defined; but many theorists prefer to use differential quantities that, in the limit of 'smallness,' behave as though they were Euclidean. One reason is that a simple algebraic metric can be written for differential quantities.

In the plane, the algebraic metric is Pythagoras' theorem: $s^2 = x^2 + y^2$, describing the relationship between the length of the hypotenuse, s, and the two sides, x and y, of a right triangle. Since the plane is flat, differential quantities are not a concern.

In a sphere, the corresponding relationship would have additional terms: $s^2 = \alpha x^2 + \beta y^2 + \gamma xy$. Such a metric is certainly approachable, but in the limit of smallness, Pythagoras theorem reappears: $ds^2 = dx^2 + dy^2$, where ds, dx, and dy are *differential* lengths. This situation is much more to many people's liking. The assumption being made, of course, is that we can always choose a sufficiently *small* portion of a sphere that we can consider flat to any accuracy we desire.

Navigators use just this type of geometry when traveling across the face of our earth. For them, metrics on the order of a few km are small enough to be considered 'flat,' given that the earth has a radius of 6,400 km.

In the geometry of the sphere, the elliptic geometry, objects again can be moved without deformation (since the surface is of *uniform* curvature), and, therefore, compared in the same sorts of relationships as in the plane. There are no parallels in the sphere, however, because there are no Euclidean straight lines, and all pairs of curves that approximate lines (the so-called *great circles* whose radii equal that of the sphere itself) always meet at two antipodal points. Neither is there a Cartesian coordinate system in the sphere. Coordinate systems in the sphere can be constructed using great circles, but these systems have no unique origin.

An *egg* is another example of an elliptic 2-space. It has two dimensions, is closed and finite. It has a differential metric like the sphere. Unlike the sphere, the egg cannot support relationships like similarity or congruence since objects cannot be moved without deformation (except for some special cases; the egg is *differentially* curved in one direction but not the other). Local coordinate systems are possible in the egg, at least over regions small enough that variations in curvature can be ignored. But a global system, like that of the sphere, is not entirely tractable.

There are higher dimensional analogs of the plane, the sphere, and the egg, and of any number of other shapes that might happen to come to mind (including the *saddle* of hyperbolic geometry). Each comprises a mathematical space in terms of being a point set with certain specially defined characteristics. In each space, different kinds of coordinate systems are possible. In the plane, we spoke of the Cartesian system; but there is also the polar system, the triangular system, and so on. All of these systems can be used to map the same plane; yet, all are different.

Physical quantities existing in the plane must be independent of the particular coordinate system chosen. These quantities are not necessarily independent of the space that contains them, however. The same idea applies to all other spaces and coordinate systems as well.

Any triangle in the plane has the property that the sum of its interior angles adds to 180°. Not so, the sphere. Any triangle in the sphere has the property that the sum of its interior angles is greater than 180°, the more-so the larger the triangle. Consider a triangle on the earth comprising one-quarter of the equator with two more legs extending toward and meeting at one of the poles. This triangle has three right angles for interior angles, giving a grand total of 270°!

Tensor analysis takes account of coordinate independence and of the peculiarities of different kinds of spaces in one grand sweep. Its formalisms are structurally the same regardless of the space involved, the number of dimensions, and so on. For this reason, tensors are very effective tools in the hands of theorists working in advanced studies. For this same reason, tensors are also very effective tools for setting up systems of equations in “everyday” physics or engineering applications. The systems themselves may not be easy to solve, but they are usually obtained with expedience.

Coordinate Curves and Coordinate Surfaces

Let’s now return to Euclidean space and consider the idea of coordinate systems a little more closely. What we learn here can be immediately extended to other types of spaces and/or to higher numbers of dimensions.

We begin with a 2-dimensional Cartesian system in a Euclidean space. The system consists of an x- and a y-axis that are orthogonal. These two axes determine a unique point of intersection. This point is designated the *origin* of the system and is given the special label $x = 0$, $y = 0$. Whole numbers are then placed along each of the axes by establishing a unit interval and using it repeatedly to mark off additional intervals.

The two axes determine a unique plane, the xy-plane. This plane is called a *coordinate surface*. Any point P in this surface can be designated by a pair of numbers, one from each of the two axes, in the following manner: Through P, two lines are constructed parallel to the individual axes. (These lines are often referred to as a *local coordinate system* or *local axes* at P.) The numerical values, $x = x_0$ and $y = y_0$, on the Cartesian axes where these lines intersect designate the *coordinates* of the point. The shorthand notation is

$$P = (x_0, y_0).$$

In a 3-dimensional Cartesian system, there are three orthogonal axes (x, y, and z) and three coordinate planes (xy, xz, and yz). Any point P is uniquely specified by the number triple

$$P = (x, y, z).$$

In an n-dimensional Cartesian system, by extension, there are n orthogonal axes and (n-1)! coordinate planes. Any point P is uniquely specified by a number *n-tuple*

$$P = (x_1, x_2, x_3, \ldots , x_n)$$

where the change to subscripted notation in necessitated for purposes of generality.

If we carefully consider the Cartesian system, we observe some specific characteristics:

- The coordinate axes are straight lines defined to intersect at a single point, the origin.
- The coordinate axes are mutually orthogonal.
- The coordinate planes are completely determined by the axes.

Suppose we were to relax these conditions. We would obtain statements to the effect that:

- The coordinate axes are general curves defined to intersect at least once. A point of intersection can be chosen as the origin.
- The coordinate axes are not *necessarily* mutually orthogonal.
- Pairs of coordinate axes uniquely determine curvilinear coordinate surfaces as *product spaces*.

This last expression needs clarification. We will proceed by example. First, consider the case of the Cartesian axes above. If one axis is *slid* along another, a unique plane is swept out. The plane is said to be the *product space* of the two lines. The coordinates of points in this plane are intuitively determined in the process, for the sliding axis changes position on the stationary axis (i.e., assumes different coordinate values from the stationary axis), and the moving axis itself has coordinates marked off on it.

Next, consider a straight line and a circle. Let the circle touch the line so that its radius is perpendicular to the line and the line is perpendicular to the plane of the circle. Now *slide* the circle along the full extent of the line. The result is that a *cylinder* will be swept out. The cylinder is the product space of the circle and the line in the configuration specified. If coordinates are marked on the line and the circle, then a unique pair of numbers will specify every point in the cylinder.

Now, consider two circles of equal radius. Let the circles be perpendicular to one another so that one circle touches the other at each of the opposite ends of a diameter. Again, sweep one of the circles around the other to produce a sphere. The sphere is the product space of the two circles in the configuration specified. If coordinates are marked on each of the circles, then a pair of numbers will uniquely specify every point in the sphere.

Similarly, a torus is the product space of two circles (not necessarily of equal radius) in a different configuration. And so on...

These last exercises describe the basis for forming non-Cartesian coordinate systems in Euclidean 3-space: i.e., the cylindrical, the spherical, and the toroidal coordinate systems.

Finally, start with *any* two curves. Let the curves intersect at one point. Mark the curves with coordinates, analogously to the coordinates on a Cartesian axis. Slide one curve along the other to produce a surface. Then a pair of numbers from the curves will specify any point on the surface in perfect analogy with the Cartesian plane. If the curves are called u and v, we then say that we have a u-axis and a v-axis. These axes together produce a uv-surface (a coordinate surface) as a product space. Any point P in this surface is specified by a pair of numbers, $u = u_0$ and $v = v_0$:

$$P = (u_0, v_0).$$

Such a system is called a *generalized* (or *curvilinear*) coordinate system. We need not limit this discussion to Euclidean spaces, for the technique described can be carried almost directly into non-Euclidean spaces as well. We will stay concerned with Euclidean spaces for the remainder of this article.

Often in physics or engineering, such systems are necessary to solve problems. For example, a NASA engineer whom I know was solving fluid dynamic equations for airflow over aircraft compressor blades. He chose to let the blade surfaces themselves represent coordinate surfaces and specified coordinate axes to fit. He wrote tensor equations in this somewhat complicated system and produced beautiful theoretical flow patterns, some of which are still hanging framed in our Administration Building and in other places around the laboratory!

Covariance and Contravariance

Writing vector or tensor equations in generalized coordinate systems is a process familiar to students in classical mechanics. In order to successfully write such equations and use them to solve problems or to build models, the characteristics of generalized coordinate systems must be understood. Recall that in a generalized coordinate system:

- The coordinate axes are general curves – we will call them u, v, w, ... , a, b, c, ...
- The coordinate axes are not *necessarily* orthogonal.
- Pairs of coordinate axes uniquely determine curvilinear surfaces as *product spaces.* These surfaces are the coordinate surfaces of the system.

Additionally:

- We can specify *local coordinate axes* at any point P in the system just as we can specify local Cartesian axes at any point in a Cartesian system.
- Similarly, we can specify *local coordinate surfaces* at any point P in the system.
- We can use the local coordinate curves and the local coordinate surfaces to specify unique sets of unit vectors at P.
- We can write any vector quantity **V** at P as a linear combination of these local unit vectors.

Now, some imagination is required. Let's return to the 3-dimensional Cartesian system. At any point P, we can specify three local axes and three local planes determined by these axes. In accordance with strict definitions, the axes must be mutually perpendicular and, by extension, so must the planes. Now, choose three unit vectors at P such that each vector is *tangent* to one of the axes. Such a triple is usually designated (**i**, **j**, **k**). Any vector **V** at P can then be written

$$\mathbf{V} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$$

where α, β, and γ are the usual x, y, and z scalar components of the vector.

Now suppose that we had chosen unit vectors *perpendicular to each of the planes* rather than *tangent to each of the coordinate axes*. Let's do so and call the resulting triple (**i***, **j***, **k***). Again, any vector **V** at P can be written

$$\mathbf{V} = \alpha^*\mathbf{i}^* + \beta^*\mathbf{j}^* + \gamma^*\mathbf{k}^*$$

where α^*, β^*, and γ^* are the scalar components of the vector *referred* to the **i***, **j***, **k*** triple. There is nothing surprising in what we have just done, and our representation is satisfactory provided we ensure that

$$\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} = \alpha^*\mathbf{i}^* + \beta^*\mathbf{j}^* + \gamma^*\mathbf{k}^*.$$

But, you might argue that what we have done is trivial since it is apparent from geometry that the two unit vector triples comprise the same set; i.e., that

$$\mathbf{i} = \mathbf{i}^*$$
$$\mathbf{j} = \mathbf{j}^*$$
$$\mathbf{k} = \mathbf{k}^*.$$

Still, we used two *distinct* approaches to defining a unit vector triple at P. Should we expect these approaches to produce so tidy a result in all cases? The answer is very definitely "*NO"!*

To understand why the answer is "NO," let's *modify* our Cartesian system so that the axes are no longer mutually orthogonal – for example, so that they meet at 60°. In this case, the origin lies at a vertex of a tetrahedron, and the axes lie along three of the edges. (Such coordinate systems are actually used in engineering and crystallography and are called *triangular* coordinate systems.) It should be intuitive that (**i**, **j**, **k**) and (**i***, **j***, **k***) are now two *different* sets of unit vectors. Specifically, **i** and **i*** now meet at an angle of 60°, as do **j** and **j***, and **k** and **k***. Thus, while they are all unit vectors, they specify *different sets of directions*, and the choice of which set to use in a given calculation must be a matter of expediency.

In tensor analysis, the same logic must be applied in generalized coordinate systems. At any point P in a generalized system, with an associated set of local axes and coordinate surfaces, we can specify two related but distinct sets of unit vectors: (1.) a set tangent to the local axes, and (2.) another set perpendicular to the local coordinate surfaces. The first set is given the name *contravariant*; the second set is given the name *covariant*. The vector **V** can be *referred* to either set, and is called *contravariant* when referred to the contravariant unit vectors or *covariant* when referred to the covariant unit vectors. As before, the choice of which to use is strictly a matter of expediency. The vector **V** is obviously not affected by the choice.

Reciprocal Sets of Vectors

Let's return to the 3-dimensional Cartesian system of our previous discussion. The unit vectors (**i**, **j**, **k**) are a contravariant set. The unit vectors (**i***, **j***, **k***) are a covariant set. The vector **V** has the contravariant representation

$$\mathbf{V} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}.$$

It also has the covariant representation

$$\mathbf{V} = \alpha^*\mathbf{i}^* + \beta^*\mathbf{j}^* + \gamma^*\mathbf{k}^*.$$

Since both representations designate the same vector **V**, we must have

$$\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} = \alpha^*\mathbf{i}^* + \beta^*\mathbf{j}^* + \gamma^*\mathbf{k}^*.$$

Let's further explore the last relationship. First of all, we know that in this special case,

$$\mathbf{i} = \mathbf{i}^*$$
$$\mathbf{j} = \mathbf{j}^*$$
$$\mathbf{k} = \mathbf{k}^*$$

so that

$$\mathbf{i}\cdot\mathbf{i}^* = \mathbf{i}^*\cdot\mathbf{i} = 1 \qquad \mathbf{j}\cdot\mathbf{i}^* = \mathbf{i}^*\cdot\mathbf{j} = 0 \qquad \mathbf{k}\cdot\mathbf{i}^* = \mathbf{i}^*\cdot\mathbf{k} = 0$$
$$\mathbf{i}\cdot\mathbf{j}^* = \mathbf{j}^*\cdot\mathbf{i} = 0 \qquad \mathbf{j}\cdot\mathbf{j}^* = \mathbf{j}^*\cdot\mathbf{j} = 1 \qquad \mathbf{k}\cdot\mathbf{j}^* = \mathbf{j}^*\cdot\mathbf{k} = 0$$
$$\mathbf{i}\cdot\mathbf{k}^* = \mathbf{k}^*\cdot\mathbf{i} = 0 \qquad \mathbf{j}\cdot\mathbf{k}^* = \mathbf{k}^*\cdot\mathbf{j} = 0 \qquad \mathbf{k}\cdot\mathbf{k}^* = \mathbf{k}^*\cdot\mathbf{k} = 1$$

Making a change in notation will help us to summarize this relationship very succinctly. Let

$$\mathbf{i} = \mathbf{u}_1$$
$$\mathbf{j} = \mathbf{u}_2$$
$$\mathbf{k} = \mathbf{u}_3$$
$$\mathbf{i}^* = \mathbf{u}_1^*$$
$$\mathbf{j}^* = \mathbf{u}_2^*$$
$$\mathbf{k}^* = \mathbf{u}_3^*.$$

Then

$$\mathbf{u}_i\cdot\mathbf{u}_j^* = \mathbf{u}_j^*\cdot\mathbf{u}_i = [1 \text{ when } i = j] \text{ or } [0 \text{ when } i \neq j].$$

The vector sets $\mathbf{u}_i$ and $\mathbf{u}_j^*$ are called, by definition, *reciprocal vector sets*. If we set $\delta_{ij} = [1$ when $i = j]$ or $[0$ when $i \neq j]$, then we can write

$$\mathbf{u}_i\cdot\mathbf{u}_j^* = \mathbf{u}_j^*\cdot\mathbf{u}_i = \delta_{ij}.$$

δ_{ij} is a component of a second rank tensor called Kronecker's delta after the mathematician Leopold Kronecker (1923-91) who first inaugurated its use. All vector sets satisfying this relationship are called reciprocal. The covariant and contravariant unit vector sets in all systems will always be (or, more generally, can always be *chosen* to be) reciprocal vector sets.

Covariant and Contravariant Base Vectors: A More General Case Study

In the case just studied, the reciprocal sets comprised *unit* vectors. In the general case, the requirement for *unit* vectors is usually dropped and replaced with a requirement for *base* vectors. It turns out that the set of relationships contained in

$$\mathbf{u}_i\cdot\mathbf{u}_j^* = \mathbf{u}_j^*\cdot\mathbf{u}_i = \delta_{ij}$$

still hold with the provision that the magnitudes of the given pairs whose inner product is unity are *reciprocal quantities*. Let's consider a generalized 3-dimensional coordinate system, u-v-w, in a Euclidean 3-space. We can refer the u-, v-, and w- axes to a Cartesian x-y-z system in the same space by transformation equations of the form

$$u = u(x, y, z) \qquad x = x(u, v, w)$$
$$v = v(x, y, z) \qquad y = y(u, v, w)$$
$$w = w(x, y, z) \qquad z = z(u, v, w)$$

as is done in basic calculus and analytic geometry. We require that the functions u, v, and w be *linearly independent* and that x, y, and z also be *linearly independent*. Thus, no one coordinate axis in either system can be written as a linear combination of the other two, and the system is truly 3-dimensional.

We can then choose a point P in the system, and specify coordinate curves and surfaces in both coordinate systems. In the generalized coordinate system, we can specify a contravariant basis set as

$$\mathbf{e}^{(1)} = \partial\mathbf{r}/\partial u,\ \mathbf{e}^{(2)} = \partial\mathbf{r}/\partial v, \text{ and } \mathbf{e}^{(3)} = \partial\mathbf{r}/\partial w$$

where, by convention, the contravariant vectors are superscripted rather than subscripted, and the vector **r** is simply the position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

in the Cartesian system. *Please note*: The parentheses around the superscripts indicate "which" base vector is being referred to; they do *not* denote tensor notation. We can also specify a covariant basis set as

$$\mathbf{e}_{(1)} = \nabla u,\ \mathbf{e}_{(2)} = \nabla v, \text{ and } \mathbf{e}_{(3)} = \nabla w$$

where, again by convention, the covariant vectors are shown subscripted[3]. Both sets are basis sets; neither set necessarily comprises unit vectors; and the two sets are reciprocal. To see the reciprocity, we must form the individual inner products:

$$\begin{aligned}\mathbf{e}^{(1)} \cdot \mathbf{e}_{(1)} &= (\partial\mathbf{r}/\partial u) \cdot (\nabla u) = (\partial x/\partial u)(\partial u/\partial x) + (\partial y/\partial u)(\partial u/\partial y) + (\partial z/\partial u)(\partial u/\partial z) \\ &= \partial u/\partial u \\ &= 1\end{aligned}$$

$$\begin{aligned}\mathbf{e}^{(1)} \cdot \mathbf{e}_{(2)} &= (\partial\mathbf{r}/\partial u) \cdot (\nabla v) = (\partial x/\partial u)(\partial v/\partial x) + (\partial y/\partial u)(\partial v/\partial y) + (\partial z/\partial u)(\partial v/\partial z) \\ &= \partial v/\partial u \\ &= 0\end{aligned}$$

and so on. The partial derivatives and chain rule used above should be familiar from basic calculus.

We can write the vector **V** in its contravariant and its covariant forms as follows:

$$\mathbf{V} = v^1\mathbf{e}^{(1)} + v^2\mathbf{e}^{(2)} + v^3\mathbf{e}^{(3)} = v_1\mathbf{e}_{(1)} + v_2\mathbf{e}_{(2)} + v_3\mathbf{e}_{(3)}.$$

If we now wish to find the magnitude of **V**, we can form the inner product $\mathbf{V} \cdot \mathbf{V}$. If, further, we use both the contravariant and the covariant representations of **V** and take advantage of the reciprocity between the two different sets of base vectors, we obtain a particularly nice result:

$$\begin{aligned}\mathbf{V} \cdot \mathbf{V} &= (v^1\mathbf{e}^{(1)} + v^2\mathbf{e}^{(2)} + v^3\mathbf{e}^{(3)}) \cdot (v_1\mathbf{e}_{(1)} + v_2\mathbf{e}_{(2)} + v_3\mathbf{e}_{(3)}) \\ &= v^1v_1 + v^2v_2 + v^3v_3 \\ &= V^2\end{aligned}$$

[3] The covariant representation retains its use of subscripts while the contravariant representation *switches from subscripts to superscripts*. This change in notation helps particularly in the case of *mixed tensors* (resulting from certain types of dyads and higher order products) where some of the vectors comprising the product **UVW...ABC** are covariant while the rest are contravariant.

The summation involves both contravariant and covariant indices. The shorthand for this process is

$$\mathbf{V} \cdot \mathbf{V} = \mathbf{\Sigma_i}\ v^i v_i\ (\textit{also} = \mathbf{\Sigma_i}\ v_i v^i) = V^2.$$

From here on, we will always take advantage of the reciprocity between the contravariant and the covariant base vector sets when constructing sums of the type given above. *Summations will always be done over a contravariant-covariant pair of indices*.

The Cartesian Fundamental Tensor

Let's again return to the 3-dimensional Cartesian system of our previous discussion. Please recall that the unit vectors (**i**, **j**, **k**) are a contravariant set, and that the unit vectors (**i***, **j***, **k***) are a covariant set. Recall also that the vector **V** has the contravariant representation

$$\mathbf{V} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$$

and the covariant representation

$$\mathbf{V} = \alpha^*\mathbf{i}^* + \beta^*\mathbf{j}^* + \gamma^*\mathbf{k}^*.$$

This time, we will again use these results to solve for the covariant components of **V** *in terms of* its contravariant components (or vice-verse).

To begin, in the equation

$$\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} = \alpha^*\mathbf{i}^* + \beta^*\mathbf{j}^* + \gamma^*\mathbf{k}^*$$

let's form the inner product of both sides of the equation by any of the six different unit vectors and again make use of their reciprocity. If we (arbitrarily) choose the unit vector **j*** with which to do this operation, we acquire

$$\left[\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} = \alpha^*\mathbf{i}^* + \beta^*\mathbf{j}^* + \gamma^*\mathbf{k}^*\right] \cdot \mathbf{j}^*$$

or

$$\alpha(\mathbf{i} \cdot \mathbf{j}^*) + \beta(\mathbf{j} \cdot \mathbf{j}^*) + \gamma(\mathbf{k} \cdot \mathbf{j}^*) = \alpha^*(\mathbf{i}^* \cdot \mathbf{j}^*) + \beta^*(\mathbf{j}^* \cdot \mathbf{j}^*) + \gamma^*(\mathbf{k}^* \cdot \mathbf{j}^*)$$

We already know how to deal with the left-hand side of this equation. The right-hand side is taken care of when we recognize that

$$\mathbf{i}^* \cdot \mathbf{i}^* = 1 \qquad \mathbf{j}^* \cdot \mathbf{i}^* = \mathbf{i}^* \cdot \mathbf{j}^* = 0 \qquad \mathbf{k}^* \cdot \mathbf{i}^* = \mathbf{i}^* \cdot \mathbf{k}^* = 0$$

$$\mathbf{i}^* \cdot \mathbf{j}^* = \mathbf{j}^* \cdot \mathbf{i}^* = 0 \qquad \mathbf{j}^* \cdot \mathbf{j}^* = 1 \qquad \mathbf{k}^* \cdot \mathbf{j}^* = \mathbf{j}^* \cdot \mathbf{k}^* = 0$$

$$\mathbf{i}^* \cdot \mathbf{k}^* = \mathbf{k}^* \cdot \mathbf{i}^* = 0 \qquad \mathbf{j}^* \cdot \mathbf{k}^* = \mathbf{k}^* \cdot \mathbf{j}^* = 0 \qquad \mathbf{k} \cdot \mathbf{k}^* = 1.$$

As before, we again let

$$\mathbf{i}^* = \mathbf{u_1}^*$$
$$\mathbf{j}^* = \mathbf{u_2}^*$$
$$\mathbf{k}^* = \mathbf{u_3}^*.$$

Then

$$\mathbf{u_i}^* \cdot \mathbf{u_j}^* = \mathbf{u_j}^* \cdot \mathbf{u_i}^* = g_{ij} = \delta_{ij}$$

where g_{ij} is a component of a second rank tensor called the *fundamental tensor*, which, in this case, just happens to be equal to δ_{ij} (and can be called the *Cartesian* fundamental tensor). In the general case, the last equality does *not* hold.

Using these relationships in the example above, we find that

$$\beta = \beta^*.$$

In this case, the same type of equality holds for the other vector components as well.

Let's see what happens when we use the base vectors $\mathbf{e}^{(i)}$, and $\mathbf{e}_{(j)}$ defined above. Again, we have

$$\mathbf{V} = v^1\mathbf{e}^{(1)} + v^2\mathbf{e}^{(2)} + v^3\mathbf{e}^{(3)} = v_1\mathbf{e}_{(1)} + v_2\mathbf{e}_{(2)} + v_3\mathbf{e}_{(3)}.$$

When we wanted to find the magnitude of **V**, we formed the inner product $\mathbf{V} \cdot \mathbf{V}$ using both the contravariant and the covariant representations of **V**. Suppose that we had just chosen one or the other. Suppose we had chosen the contravariant representation. Then

$$\begin{aligned} V^2 = \mathbf{V} \cdot \mathbf{V} &= (v^1\mathbf{e}^{(1)} + v^2\mathbf{e}^{(2)} + v^3\mathbf{e}^{(3)}) \cdot (v^1\mathbf{e}^{(1)} + v^2\mathbf{e}^{(2)} + v^3\mathbf{e}^{(3)}) \\ &= (v^1\mathbf{e}^{(1)})\cdot(v^1\mathbf{e}^{(1)}) + (v^1\mathbf{e}^{(1)})\cdot(v^2\mathbf{e}^{(2)}) + (v^1\mathbf{e}^{(1)})\cdot(v^3\mathbf{e}^{(3)}) + \\ &\quad (v^2\mathbf{e}^{(2)})\cdot(v^1\mathbf{e}^{(1)}) + (v^2\mathbf{e}^{(2)})\cdot(v^2\mathbf{e}^{(2)}) + (v^2\mathbf{e}^{(2)})\cdot(v^3\mathbf{e}^{(3)}) + \\ &\quad (v^3\mathbf{e}^{(3)})\cdot(v^1\mathbf{e}^{(1)}) + (v^3\mathbf{e}^{(3)})\cdot(v^2\mathbf{e}^{(2)}) + (v^3\mathbf{e}^{(3)})\cdot(v^3\mathbf{e}^{(3)}) \\ &= g_{11}(v^1)^2 + g_{12}\,v^1v^2 + g_{13}\,v^1v^3 + \\ &\quad g_{21}\,v^2v^1 + g_{22}\,(v^2)^2 + g_{23}\,v^2v^3 + \\ &\quad g_{31}\,v^3v^1 + g_{32}\,v^3v^2 + g_{33}\,(v^3)^2. \end{aligned}$$

The shorthand for this process is

$$\mathbf{V} \cdot \mathbf{V} = \mathbf{\Sigma_i \Sigma_j}\; g_{ij}\, v^i v^j = V^2.$$

While this relationship is perfectly correct, it lacks the simplicity of the previous relationship in that it involves both a double summation and extra terms (the g_{ij}). A similar argument can be made for using just the covariant indices.

Reciprocal Sets and the Fundamental Tensor in the General Case

These arguments can be extended directly to the case of a vector in a non-orthogonal, non-linear n-dimensional coordinate system. For the moment, let's just stick to a 3-dimensional system. The extension to n-dimensions should be intuitively clear.

Using local coordinate axes and surfaces at the point P, let a vector **V** at P again be represented in two different ways:

$$\mathbf{V} = \alpha^1\mathbf{e}^{(1)} + \alpha^2\mathbf{e}^{(2)} + \alpha^3\mathbf{e}^{(3)} = \alpha_1\mathbf{e}_{(1)} + \alpha_2\mathbf{e}_{(2)} + \alpha_3\mathbf{e}_{(3)}.$$

Notice that the same change in notation as before has again been introduced. Since the covariant and contravariant basis sets are reciprocal sets, we must have

$$\mathbf{e}^{(i)} \cdot \mathbf{e}_{(j)} = \mathbf{e}_{(j)} \cdot \mathbf{e}^{(i)} = \delta_i^j.$$

Notice that Kronecker's delta is now a component of a *mixed tensor of rank 2* in the general case; i.e., the *index* (superscript) j is a *contravariant* index, while the index (subscript) i is a *covariant* index. (Notice also that the superscript 'i' in the inner products becomes a covariant index in the delta, and that the subscript 'j' in the inner products becomes a contravariant index in the delta. Again, recall that the letters in parentheses are not tensorial indices.)

Next let's write out the fundamental tensor in its covariant and contravariant forms:

$$\mathbf{e}^{(i)} \cdot \mathbf{e}^{(j)} = g_{ij} \text{ (covariant) and } \mathbf{e}_{(i)} \cdot \mathbf{e}_{(j)} = g^{ij} \text{ (contravariant).}$$

(Notice again that the covariant and contravariant fundamental tensors arise from the superscripted and subscripted sets of unit vectors, respectively. Now the going becomes more direct. These last two are the only cases in which this peculiar switch takes place.)

Now, let's solve for the contravariant components of **V** in terms of the covariant components. We will introduce yet another shorthand notation, the so-called *Einstein summation convection*. Notice that **V** can be written as

$$\mathbf{V} = \mathbf{\Sigma_i}\ \alpha^i\ \mathbf{e^{(i)}} = \mathbf{\Sigma_j}\ \alpha_j\ \mathbf{e_{(j)}}.$$

Einstein noticed that summation always occurs over a *repeated* index so that it is not strictly necessary to write out the summation operator ('$\mathbf{\Sigma_i}$'or '$\mathbf{\Sigma_j}$') each and every time. Using this convention, we have the compact notation

$$\mathbf{V} = \alpha^i\ \mathbf{e^{(i)}} = \alpha_j\ \mathbf{e_{(j)}}$$

where summation is now understood to occur over i in the middle and j on the right-hand side.

Using this convention and applying the results of the previous paragraphs, we first form the inner product

$$\mathbf{V} \cdot \mathbf{e_{(s)}} = \alpha^i\ (\mathbf{e^{(i)}} \cdot\ \mathbf{e_{(s)}}) = \alpha_j\ (\mathbf{e_{(j)}} \cdot \mathbf{e_{(s)}})$$

where a new index 's' is introduced because no summation is intended. Next, we notice that the middle term reduces to

$$\alpha^i\ (\mathbf{e^{(i)}} \cdot \mathbf{e_{(s)}}) = \alpha^i\ \delta_i^{\ s} = \alpha^s$$

as the reader can show by writing out the expression in full, and that the right-hand terms reduce to

$$\alpha_j\ (\mathbf{e_{(j)}} \cdot \mathbf{e_{(s)}}) = \alpha_j\ g^{js}.$$

Thus, we conclude that

$$\alpha^s = \alpha_j\ g^{js}$$

where the summation on the right-hand side is over the index 'j'. Thus, the contravariant components of **V** are linear combinations of the covariant components. Expanded, this same expression becomes

$$\begin{aligned}
\alpha^1 &= \alpha_1\ g^{11} + \alpha_2\ g^{21} + \alpha_3\ g^{31} \\
\alpha^2 &= \alpha_1\ g^{12} + \alpha_2\ g^{22} + \alpha_3\ g^{32} \\
\alpha^3 &= \alpha_1\ g^{13} + \alpha_2\ g^{23} + \alpha_3\ g^{33}
\end{aligned}$$

Notice that the free index 's' takes on sequential values (1, 2, 3) while the repeated index 'j' represents summation for *each* new value of 's.'

Similarly, the covariant components can be expressed as linear combinations of the contravariant components

$$\alpha_r = \alpha^i\ g_{ir}$$

where the summation on the right-hand side is over the index 'i.'

This little exercise is typical of beginning tensor analysis. It not only shows the index notation in full swing, but also provides two relationships fundamental to tensor algebra, specifically

$$\alpha^s = \alpha_j \, g^{js}$$

and

$$\alpha_r = \alpha^i \, g_{ir}.$$

One final note on the fundamental tensor is that it is *symmetric*: i.e., $g_{ij} = g_{ji}$, and $g^{st} = g^{ts}$. To understand why this is so, let's return to the definitions

$$\mathbf{e^{(i)} \cdot e^{(j)}} = g_{ij} \text{ and } \mathbf{e_{(i)} \cdot e_{(j)}} = g^{ij}.$$

We know from our basic vector analysis that the inner product is symmetric, i.e., that

$$\mathbf{e^{(i)} \cdot e^{(j)} = e^{(j)} \cdot e^{(i)}} \text{ and } \mathbf{e_{(i)} \cdot e_{(j)} = e_{(j)} \cdot e_{(i)}}.$$

From this, the symmetry of the fundamental tensor follows directly.

Coordinate Transformations: Scalars

We now have the mathematical tools that enable us to achieve our stated objective of approaching Line 1, Page 1 of any standard text on tensor analysis. These texts typically begin by stating that tensors obey particular transformation laws whose forms are then specified with little or no motivating arguments. We have already looked at tensors as representing physical quantities and agreed that such quantities must appear the same to all observers. If we now formally replace the word 'observer' with the word 'coordinate system' we can paraphrase our previous statement to read that tensors are quantities that must be invariant under a coordinate transformation; i.e., they must retain a certain character no matter how we look at them. Their components might vary from system to system, but their overall structure must remain the same.

Let's now consider the simplest case, the scalar. Let S be any scalar quantity observed in a system K. Let S* be the same scalar quantity observed in a system K*. Then, for S to represent a real physical quantity, we must require that

$$S = S^*.$$

Any quantity satisfying this transformation law is called a tensor of rank 0. This concludes our study of scalars.

Coordinate Transformations: Vectors

Let's next consider a vector. Let **V** be a vector quantity observed in a system K and **V*** be the same vector quantity observed in K*. Then, for **V** to represent a real physical quantity, we must require that

$$\mathbf{V} = \mathbf{V^*}.$$

Any quantity satisfying this transformation law is called a tensor of rank 1. In order to derive transformation laws, we can represent **V** in each system in either its covariant or contravariant form. Let's use the base vector system introduced earlier and consider the covariant case in some detail.

First, let's review: Recall that we had a generalized 3-dimensional coordinate system u-v-w in a Euclidean 3-space. Let this system now become the system K above.

We referred the u-, v-, and w- axes to a Cartesian x-y-z system in the same space by transformation equations of the form

$$u = u(x, y, z) \qquad x = x(u, v, w)$$
$$v = v(x, y, z) \qquad y = y(u, v, w)$$
$$w = w(x, y, z) \qquad z = z(u, v, w)$$

We now change this notation as follows:

$$u = u^1 \qquad x = x^1$$
$$v = u^2 \qquad y = x^2$$
$$w = u^3 \qquad z = x^3$$

so that the coordinate transformation equations reduce to

$$u^i = u^i\,(x^j) \qquad\qquad x^s = x^s\,(u^t)$$

with i, j, s, t, = 1, 2, 3. We then chose a point P in the system, and specified coordinate curves and surfaces at P. In the generalized coordinate system, we specified a contravariant basis set as

$$\mathbf{e^{(1)}} = \partial\mathbf{r}/\partial u,\ \mathbf{e^{(2)}} = \partial\mathbf{r}/\partial v, \text{ and } \mathbf{e^{(3)}} = \partial\mathbf{r}/\partial w$$

and a covariant basis set as

$$\mathbf{e_{(1)}} = \nabla u,\ \mathbf{e_{(2)}} = \nabla v, \text{ and } \mathbf{e_{(3)}} = \nabla w.$$

Using the new notation just introduced, these expressions become

$$\mathbf{e^{(i)}} = \partial\mathbf{r}/\partial u^i \text{ and } \mathbf{e_{(j)}} = \nabla u^j.$$

<u>*Now, let's move on*</u>: We can make identical specifications for K*, first with coordinate transformations

$$u^i{*} = u^i{*}\,(x^j) \qquad\qquad x^s = x^s\,(u^t{*})$$

and then with base vectors

$$\mathbf{e^{(i)}*} = \partial\mathbf{r}/\partial u^i{*} \text{ and } \mathbf{e_{(j)}*} = \nabla u^j{*}.$$

Now, using the covariant representation, the expression $\mathbf{V} = \mathbf{V}^*$ then becomes

$$\mathbf{V} = V_i\,\mathbf{e^{(i)}} = V_j{*}\,\mathbf{e^{(j)}*} = \mathbf{V}^*$$

The middle of the equation expands to read

$$V_i\,\nabla u^i = V_j{*}\,\nabla u^j{*}$$

But[4] $\nabla u^j{*} = [\partial u^j{*}/\partial x^s] = (\partial u^j{*}/\partial u^k)\,[\partial u^k/\partial x^s] = (\partial u^j{*}/\partial u^k)\,\nabla u^k$. Substituting, we acquire

$$V_i\,\nabla u^i = V_j{*}\,(\partial u^j{*}/\partial u^k)\,\nabla u^k$$

or, changing summation indices on the right-hand side

$$V_i\,\nabla u^i = V_j{*}\,(\partial u^j{*}/\partial u^i)\,\nabla u^i.$$

Finally, we write

$$(V_i - V_j{*}\,(\partial u^j{*}/\partial u^i))\,\nabla u^i = 0$$

and, since $\nabla u^i \neq 0$ necessarily, we must have

[4] **NOTE**: We are using the shorthand notation $[\partial u^j{*}/\partial x^s] = [\partial u^j{*}/\partial x^1,\ \partial u^j{*}/\partial x^2,\ \partial u^j{*}/\partial x^3]$ to represent the gradient of u^j.

$$V_i = V_j^* (\partial u^{j*}/\partial u^i)$$

with summation over the index j. This expression is the *general transformation law for a covariant tensor of rank 1*. A similar calculation can be completed for the transformation of the contravariant components. The result is

$$V^i = V^{j*} (\partial u^i/\partial u^{j*}).$$

The two equations

$$V_i = V_j^* (\partial u^{j*}/\partial u^i) \text{ and } V^i = V^{j*} (\partial u^i/\partial u^{j*})$$

are usually taken as definitions of covariant and contravariant tensors of rank 1. Textbooks often begin at this point.

Epilogue

If you had difficulty following the last few arguments, please take time to write out the summations in full. Doing so will require some care in mathematical bookkeeping but will be well worth the effort. Once the technique is mastered, it can then be repeated for higher order vector products such as dyads, triads, or the general "n-ad" product **UVW...ABC**. You are urged to try. From this point on, you should also be able to work your way through any basic text in the subject.

Many important theories in physics are developed using tensors because of their succinctness and their relative "ease" in utility. The resulting algebraic and/or differential equations can be extremely difficult to solve, but the procedure for arriving at them is usually very direct. Among the most famous of the theoretical developments using tensors are the theory of electromagnetism and the special and general theories of relativity.

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13. ABSTRACT ***(Maximum 200 words)***

Tensor analysis is the type of subject that can make even the best of students shudder. My own post-graduate instructor in the subject took away much of the fear by speaking of an implicit *rhythm* in the peculiar notation traditionally used, and helped us to see how this rhythm plays its way throughout the various formalisms. Prior to taking that class, I had spent many years "playing" on my own with tensors. I found the going to be tremendously difficult but was able, over time, to back out some physical and geometrical considerations that helped to make the subject a little more transparent. Today, it is sometimes hard not to think in terms of tensors and their associated concepts. This article, prompted and greatly enhanced by Marlos Jacob, whom I've met only by e-mail, is an attempt to record those early notions concerning tensors. It is intended to serve as a bridge from the point where most undergraduate students "leave off" in their studies of mathematics to the place where most texts on tensor analysis begin. A basic knowledge of vectors, matrices, and physics is assumed. A semi-intuitive approach to those notions underlying tensor analysis is given via scalars, vectors, dyads, triads, and higher vector products. The reader must be prepared to do some mathematics and to think. For those students who wish to go beyond this humble start, I can only recommend my professor's wisdom: find the *rhythm* in the mathematics and you will fare pretty well.

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